REMARKS ON MR. MEECH'S ARTICLE ON ELLIPTIC FUNCTIONS.

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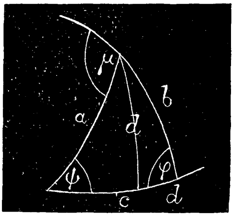
1. In Section II, the spherical triangle is used to prove the theorem of addition of the first species, but in this proof $\cos C$ is taken negative. (In this Mr. Meech agrees with all authors I have seen, for instance see Schellbach, page 106.) This has always puzzled me and I finally discovered the following to me more natural demonstration of this important theorem.



Let there be a spherical triangle whose sides are a, b, c and opposite angles φ, ψ, $\pi-\mu$ respectively, or let μ be the exterior angle for which we should have if the triangle was plane

$$\mu = \varphi+\psi, \tag{1}$$

or

$$\sin\mu = \sin\varphi\cos\psi+\cos\varphi\sin\psi. \tag{2}$$

It is required to find analogous relations for the spherical triangle. From μ let fall a perpendicular on c, then we have, if the segment adjacent to b is denoted by d,

$$\tan d = \tan b\cos\varphi,$$
$$\tan(c-d) = \tan a\cos\psi;$$

hence

$$\sin d = \frac{\tan b\cos\varphi}{\sqrt{(1+\tan^2 b\cos^2\varphi)}}; \qquad \cos d = \frac{1}{\sqrt{(1+\tan^2 b\cos^2\varphi)}};$$

$$\sin(c-d) = \frac{\tan a\cos\psi}{\sqrt{(1+\tan^2 a\cos^2\psi)}}; \quad \cos(c-d) = \frac{1}{\sqrt{(1+\tan^2 a\cos^2\psi)}};$$

consequently

$$\sin c = \frac{\tan b\cos\varphi+\tan a\cos\psi}{\sqrt{[(1+\tan^2 b\cos^2\varphi)(1+\tan^2 a\cos^2\psi)]}}. \tag{3}$$

Place

$$\frac{\sin c}{\sin\mu} = \frac{\sin b}{\sin\psi} = \frac{\sin a}{\sin\varphi} = \varepsilon; \tag{4}$$

also $\cos c = \sqrt{(1-\varepsilon^2\sin^2\mu)} = \Delta\mu;\ \cos b = \Delta\psi;\ \cos a = \Delta\varphi;$ (5)

substituting into (3) and reducing we obtain

$$\sin\mu = \frac{\sin\varphi\cos\psi\,\Delta\psi+\sin\psi\cos\varphi\,\Delta\varphi}{1-\varepsilon^2\sin^2\varphi\sin^2\psi}. \tag{2'}$$

This is analogous to (2). The following are deduced from this;

$$\cos\mu = \frac{\cos\varphi\cos\psi-\sin^2\varphi\sin\psi\,\Delta\varphi\,\Delta\psi}{1-\varepsilon^2\sin^2\varphi\sin^2\psi}; \tag{6}$$

$$\Delta\mu = \frac{\Delta\varphi\ \Delta\psi - \varepsilon^2 \sin\varphi \sin\psi \cos\varphi \cos\psi}{1 - \varepsilon^2 \sin^2\varphi \sin^2\psi}. \qquad (7)$$

We have

$$\cos\mu\, d\mu = \frac{d.\sin\mu}{d\varphi} d\varphi + \frac{d.\sin\mu}{d\psi} d\psi. \qquad (8)$$

But we obtain after reduction

$$\frac{d.\sin\mu}{d\varphi} = \frac{\cos\varphi\cos\psi - \sin\varphi\sin\psi\,\Delta\varphi\,\Delta\psi}{\Delta\varphi(1-\varepsilon^2\sin^2\varphi\sin^2\psi)^2}\left(\Delta\varphi\,\Delta\psi - \varepsilon^2\sin\varphi\sin\psi\cos\varphi\cos\psi\right)$$

$$= \frac{\cos\mu\,\Delta\mu}{\Delta\varphi};\ \text{also}\ \frac{d.\sin\mu}{d\psi} = \frac{\cos\mu\,\Delta\mu}{\Delta\psi}.$$

Hence in (8)

$$\frac{d\mu}{\Delta\mu} = \frac{d\varphi}{\Delta\varphi} + \frac{d\psi}{\Delta\psi}. \qquad (9)$$

$$\therefore \qquad F_0^{\mu}(\varepsilon) = F_0^{\phi}(\varepsilon) + F_0^{\psi}(\varepsilon); \qquad (11)$$

also

$$F_0^{\phi}(\varepsilon) = F_{\psi}^{\mu}(\varepsilon). \qquad (10)$$

If $\varepsilon = 0$ (1′) becomes the same as (1).

2. In Section III, the method of evaluating elliptics of the first species by means of ascending moduli and corresponding amplitudes is explained as Legendre and most authors have done it. I think however the following method invented by myself will be found more convenient in practice.

Suppose we have to evaluate

$$\int_0^{\phi} \frac{d\varphi}{\sqrt{(a^2 - c^2\sin^2\varphi)}} = \frac{1}{a} F_0^{\phi}\left(\frac{c}{a}\right). \qquad (10)$$

Place $a' = \frac{1}{2}(a+c)$; $c' = \sqrt{(ac)}$, and $\sin(2\varphi' - \varphi) = (c \div a)\sin\varphi$. (11)

We have then

$$\frac{1}{a} F_0^{\phi}\left(\frac{c}{a}\right) = \int_0^{\phi} \frac{d\varphi}{\sqrt{(a^2 - c^2\sin^2\varphi)}} = \int_0^{\phi'} \frac{d\varphi}{\sqrt{(a'^2 - c'^2\sin^2\varphi)}} = \frac{1}{a'} F_0^{\phi'}\left(\frac{c'}{a'}\right). \qquad (12)$$

Again let $a'' = \frac{1}{2}(a' + c')$; $c'' = \sqrt{(a'c')}$, and $\sin(2\varphi'' - \varphi') = (c' \div a')\sin\varphi'$, (11′)

then we have, proceeding in this manner

$$\frac{1}{a} F_0^{\phi}\left(\frac{c}{a}\right) = \int_0^{\phi} \frac{d\varphi}{\sqrt{(a^2 - c^2\sin^2\varphi)}} = \int_0^{\phi'} \frac{d\varphi}{\sqrt{(a'^2 - c'^2\sin^2\varphi)}} = \frac{1}{a'} F_0^{\phi'}\left(\frac{c'}{a'}\right)$$

$$= \int_0^{\phi''} \frac{d\varphi}{\sqrt{(a''^2 - c''^2\sin^2\varphi)}} = \frac{1}{a''} F_0^{\phi''}\left(\frac{c''}{a''}\right)$$

$$= \ldots\ldots\ldots\ldots$$

$$= \int_0^{\phi^{(n)}} \frac{d\varphi}{\sqrt{(a^{(n)2} - c^{(n)2}\sin^2\varphi)}} = \frac{1}{a^{(n)}} F_0^{\phi^{(n)}}\left(\frac{c^{(n)}}{a^{(n)}}\right)$$

and if $a^{(n)} = c^{(n)}$ within the accuracy of the computation,

$$\frac{1}{a}F_0^{\phi}\left(\frac{c}{a}\right) = \int_0^{\phi(n)} \frac{d\varphi}{c^{(n)}\cos\varphi} = \frac{1}{c^{(n)}} \log\tan\left(\frac{\pi}{4}+\frac{\varphi^{(n)}}{2}\right) \tag{13}$$
$$= \frac{1}{c^{(n)}} F_0^{\phi^{(n)}}(1).$$

The quantity $c^{(n)}$ is the limit reached by taking successively arithmetic and geometric means, or it may be called

the arithmetic-geometric mean of a and c,

and in ANALYST, Vol. IV, No. 5, p. 157, I have ventured on the symbol

$$\text{arithmetic-geometric mean of } a \text{ and } c = \left\| \tfrac{1}{2}\left(a \overset{+}{\underset{\times}{}} c\right)^{\frac{1}{2}} \right\|.$$

In Vol. IV, No. 4, page 121, I have exhibited the very convenient computation of such a quantity. The scale ef amplitudes being computed simultaneously the complete value is found very readily.

3. Similarly, for the descending scale of moduli the following method, first given by Gauss in 1818, may be used.

Suppose we have to evaluate

$$\frac{1}{a_0}F_0^{\phi_0}\left[\sqrt{\left(1-\frac{b_0^2}{a_0^2}\right)}\right] = \int_0^{\phi_0} \frac{d\varphi}{\sqrt{(a_0^2\cos^2\varphi+b_0^2\sin^2\varphi)}}. \tag{14}$$

Place $a_1 = \frac{1}{2}(a_0+b_0)$; $b_1 = \sqrt{(a_0 b_0)}$; $\tan(\varphi_1 - \varphi_0) = (b \div a)\tan\varphi_0$, (15)

then
$$\frac{1}{a_0}F_0^{\phi}\left[\sqrt{\left(1-\frac{b_0^2}{a_0^2}\right)}\right] = \int_0^{\phi_0} \frac{d\varphi}{\sqrt{(a_0^2\cos^2\varphi+b_0^2\sin^2\varphi)}}$$
$$= \frac{1}{2}\int_0^{\phi_1} \frac{d\varphi}{\sqrt{(a_1^2\cos^2\varphi+b_1^2\sin^2\varphi)}} = \frac{1}{2a_1}F_0^{\phi_1}\left[\sqrt{\left(1-\frac{b_1^2}{a_1^2}\right)}\right]. \tag{16}$$

Placing $a_2 = \frac{1}{2}(a_1+b_1)$; $b_2 = \sqrt{(a_1 b_1)}$; $\tan(\varphi_2 - \varphi_1) = (b_1 \div a_1)\tan\varphi_1$ (17)
we have likewise

$$\frac{1}{2a_1}F_0^{\phi_1}\left[\sqrt{\left(1-\frac{b_1^2}{a_1^2}\right)}\right] = \frac{1}{2^2 a_2}F_0^{\phi_2}\left[\sqrt{\left(1-\frac{b_2^2}{a_2^2}\right)}\right]. \tag{16'}$$

If
$$a_n = b_n = \left\| \tfrac{1}{2}\left(a_0 \overset{+}{\underset{\times}{}} b_0\right)^{\frac{1}{2}} \right\| \tag{18}$$

then
$$\frac{1}{a_n}F_0^{\phi_n}\left[\sqrt{\left(1-\frac{b_n^2}{a_n^2}\right)}\right] = \frac{1}{a_n}F_0^{\phi_n}(0) = \frac{\varphi_n}{a_n} = \varphi_n \div \left\| \tfrac{1}{2}\left(a_0 \overset{+}{\underset{\times}{}} b_0\right)^{\frac{1}{2}} \right\|. \tag{19}$$

Therefore
$$\frac{1}{a_0}F_0^{\phi_0}\left[\sqrt{\left(1-\frac{b_0^2}{a_0^2}\right)}\right] = \varphi_n \div 2^n \left\| \tfrac{1}{2}\left(a_0 \overset{+}{\underset{\times}{}} b_0\right)^{\frac{1}{2}} \right\|. \tag{20}$$

This method of evaluating is especially convenient if φ_0 is some multiple of $\frac{1}{2}\pi$, as two examples in the ANALYST prove.

I have prepared formulas by means of the same substitutions for evaluating the second and third species, which I may communicate at some future time.